

# Unconstrained canonical action for, and positive energy of, massive spin 2

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## Abstract

Filling a much-needed gap, we exhibit the  $D = 4$  Fierz-Pauli (FP) massive  $s = 2$  action, and its – manifestly positive – energy, in terms of its  $2s + 1 = 5$  unconstrained helicity  $(2,1,0)$  excitations, after reducing and diagonalizing the troublesome helicity-0 sector.

## 1 Introduction

We begin with an apologia, since there is nothing still not known about the FP model [1]. However, there seems to be no published derivation<sup>1</sup> from its covariant and highly constrained form to the final unconstrained canonical action in terms of its  $2s + 1 = 5$  helicity  $(\pm 2, \pm 1, 0)$  components. That form will display that each mode propagates correctly and has manifestly positive energy, in the usual  $L = p\dot{q} - H$ ,  $H = \frac{1}{2}[p^2 + q(-\nabla^2 + m^2)q]$  form. While [1] realized that positive energy was essential, it was displayed rather opaquely; a subsequent formulation [3] was likewise less than transparently presented (and contains distracting typos). Indeed, proper use of the constraints is not altogether trivial, making the correct process instructive as a (minor) exercise in (free) field theory.

Historically, it was not until FP's 1939 work that the problem of representing massive spins  $> 1$  by tensor fields – involving many more than  $2s + 1$  components – was raised, let alone solved. Given the current interest in massive gravity (mGR) with Einstein kinetic terms plus non-derivative mass terms involving a fixed, say flat, background, our summary may be useful. Indeed, this is a good place to note that – contrary to statements in the mGR literature – the mass terms destroy the whole (ADM) asymptotic energy formulation of GR as a  $2D$  surface integral at spatial infinity, just as the Coulomb asymptotic integral that counts total charge is lost in massive (Proca) vector

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<sup>1</sup>This was actually done [2] for the more general case of the system embedded in dS, rather than flat, space; however that derivation involved (an even number of) steps using inverse powers of  $\Lambda$ , hence singular and not applicable here; its final result of course does limit to ours. It was also shown in detail here that the helicity 0 mode can be removed by suitably tuning  $m^2/\Lambda$  in dS.

theory: the Newtonian/Coulomb fields decay much too fast there for these integrals to contribute at all.

## 2 The Derivation

The action and field equations of the theory are the sum of linearized GR and the -unique-mass term that eliminates the 6th, ghost helicity 0, degree of freedom (DoF). We work throughout in 1st order,  $3 + 1$ , canonical form, which simplifies the procedure and indeed starts directly in terms of the 6 conjugate pairs  $(\pi^{ij}, h_{ij})$  rather than the 10 covariant  $h_{\mu\nu}$ . The action is (see e.g., [2])

$$I = \int d^4x \left\{ \pi^{ij} \dot{h}_{ij} - H(\pi, h) \right\}, \quad (1)$$

$$H = {}^3R_Q + \left( \pi_{ij}^2 - \frac{1}{2} \pi^2 \right) + 4n R_0 + 2N_i \partial_j \pi^{ij} + \frac{1}{4} m^2 (h_{ij}^2 - h_{ii}^2 - 4n h_{ii} - 2N_i^2),$$

where (under the integral)

$${}^3R_Q = \frac{1}{2} h_{ij} G_{ij}^L = -\frac{1}{4} [h_{\text{TT}} \nabla^2 h_{\text{TT}} - h_{\text{T}} \nabla^2 h_{\text{T}}], \quad R_0 = (m^2 - \nabla^2) h_{\text{T}} + m^2 h_{\text{L}}; \quad (2)$$

$n \sim \frac{1}{2} h_{00}$  is a Lagrange multiplier enforcing the linear constraint  $R_0 = 0$ , while  $N_i = h_{0i}$  becomes an auxiliary field to be eliminated by completing squares, leaving only the six  $(\pi, h)$  pairs – indeed our whole process consists of juggling quadratic forms. Finally, we recall that the linearized 3D Einstein tensor  $G_{ij}^L(h_{lm})$  is both identically conserved and independent of the longitudinal, gauge, parts of  $h_{lm}$ ; The second ingredient, essential to the separation of the various helicity DoF in (1), is the usual orthogonal decomposition of any symmetric 3-tensor,

$$S_{ij} = S_{ij}^{\text{TT}} + \frac{1}{2} (\delta_{ij} - \hat{\partial}_i \hat{\partial}_j) S^{\text{T}} + [\hat{\partial}_j S_i^{\text{T}} + \hat{\partial}_i S_j^{\text{T}}] + \hat{\partial}_i \hat{\partial}_j S^{\text{L}}, \quad \partial_i S_i^{\text{T}} \equiv 0, \quad \hat{\partial}_i \equiv \partial_i / \sqrt{\nabla^2}. \quad (3)$$

Completing squares in (1) removes the  $N_i$  dependence of  $H$  in favor of adding the term  $2m^{-2}(\partial_j \pi^{ij})^2$  to  $H$ . There remains the elimination of the  $R_0$  constraint, hence of one linear combination of the two helicity 0 ( $T$ ,  $L$ ) modes. It will be equally essential to use  $\dot{R}_0 = 0$  to further eliminate one combination of their conjugate momenta  $(\pi^{\text{T}}, \pi^{\text{L}})$  using  $\dot{h} \sim \pi$ ; constraints “strike twice” in our 1st order form, since they are valid for all times<sup>2</sup>.

The task before us then is to decompose (1) into a sum of three – non- interacting – orthogonal, two DoF sectors: Helicity  $\pm 2$  (TT), helicity  $\pm 1$  “ $T_i$ ”, and the  $(T, L)$  helicity-0. The latter’s Hamiltonian is the source of difficulty, being a priori non-positive before using the  $R_0(T, L) = 0$  constraint. To keep the discussion compact, we first dispose of the helicity  $> 0$  sectors: that of TT is trivial to obtain, being unconstrained; we simply add up the TT terms in (1); dropping “TT”,

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<sup>2</sup> That the original number, 6, of  $\pi^{ij} \dot{h}_{ij}$  kinetic terms decreases by one for every constraint is just Darboux’s theorem on quadratic forms; in massless theory there are 4 constraints, leaving just the two  $(\pi^{\text{TT}}, q^{\text{TT}})$  pairs. We will see this more explicitly below.

we have

$$L = \pi^{ij} \dot{h}_{ij} - H \quad H = \pi_{ij}^2 + \frac{1}{4} (h_{ij,k}^2 + m^2 h_{ij}^2) \geq 0; \quad (4)$$

Note that  $H$  vanishes only for TT vacuum,  $\pi = 0 = h$ . The same is true of the transverse vector ( $T_i$ ) part, though it still requires some field redefinitions to achieve the same final form; here (1) also easily yields (omitting “ $T_i$ ”)

$$H = \pi^2 + 2m^{-2} \left( \pi^{ij}_{,j} \right)^2 + \frac{1}{2} m^2 h^2 \geq 0; \quad (5)$$

while not (yet) very pretty, this  $H$  is also positive and vanishes at ( $T_i$ ) vacuum, a result unaffected by the further field redefinitions required to reach the final  $p\dot{q} - H$  form; we outline the process in the Appendix. [Recall, however, that correct energy functional form is only reached when the “( $p, q$ )” variables are redefined to ensure that the associated kinetic, “ $p\dot{q}$ ”, term is itself free of unwanted numerical coefficients.]

We now face the final,  $H(T, L)$ , sector, where  $R_0$  must be used – twice. There,

$$I[T, L] = \int d^4x \left[ \frac{1}{2} \pi^T \dot{h}_T + \pi^L \dot{h}_L - V(h_T, h_L) - K(\pi^T, \pi^L) \right], \quad (6)$$

$$4V(h) \equiv h_T (\nabla^2 - m^2) h_T - 2h_T h_L, \quad K(\pi) \equiv \frac{1}{2} (\pi^L)^2 - 2\pi^T \pi^L + 2\pi^L (-m^{-2} \nabla^2) \pi^L.$$

We now show that both potential and kinetic parts of  $H$  are positive, using the  $R_0$  constraint and its time derivative respectively. Eliminating  $h_L$  yields

$$V(h_L, h_T) = \frac{3}{8} \left[ (h_{T,i})^2 + m^2 h_T^2 \right] \geq 0; \quad (7)$$

again,  $V$  only vanishes at vacuum,  $h_T = 0$ . Next we find the  $\dot{R}_0 = 0$  constraint between  $\pi^T$  and  $\pi^L$ : The two field equations for  $\pi \sim \dot{h}$  obtained by varying (6) w.r.t. the  $\pi$  are

$$\dot{h}_L - \pi^L + \pi^T + 4m^{-2} \nabla^2 \pi^L = 0, \quad \dot{h}_T + 2\pi^L = 0. \quad (8)$$

Taking their appropriate vanishing linear combination, we learn that

$$m^2 \pi^T = (-2\nabla^2 - m^2) \pi^L; \quad (9)$$

hence finally

$$K(\pi^L, \pi^T) \rightarrow K(\pi^L) = \frac{3}{2} (\pi^L)^2 \geq 0. \quad (10)$$

We have now established  $E \geq 0$  for the full theory, but one task is still to be completed: putting the helicity action into exact  $p\dot{q} - H(p, q)$  form. Even before this is done, one can already see that the (second order) field equations are uniformly  $(\square - m^2)h = 0$ , but it is an amusing exercise – as well as a check on the result – to do so. Using (9), it is easy to translate the  $(T, L)$  sector’s

$\pi^L \dot{h}_L + \frac{1}{2} \pi^T \dot{h}_T$  into  $\pi^L \dot{h}_T$  form. At this penultimate point,

$$L(T, L) = -\frac{3}{2} \pi^L \dot{h}_T - \frac{3}{2} [(\pi^L)^2 + \frac{1}{4} (h_{T,i,i})^2 + \frac{1}{4} m^2 h_T^2]; \quad (11)$$

the obvious rescaling  $(\pi, h) \rightarrow \sqrt{2/3}(-\pi, h)$  achieves the desired final canonical form of the helicity 0 sector,

$$L(0) = \pi \dot{h} - H(\pi, h), \quad H \equiv \pi^2 + \frac{1}{4} h (-\nabla^2 + m^2) h. \quad (12)$$

Together with the vector mode in the Appendix, then, the total action is

$$L = \sum_{A=1}^5 p^A \dot{q}_A - \frac{1}{2} [(p^A)^2 + q_A (-\nabla^2 + m^2) q_A], \quad (13)$$

after the (cosmetic) rescaling  $\pi \rightarrow p^A/\sqrt{2}$ ,  $h \rightarrow q_A \sqrt{2}$ .

### 3 Summary

The physical correctness of the massive  $s = 2$  FP model has been displayed: each of its  $2s + 1 = 5$  helicity excitations obey  $(\square - m^2) h = 0$ ,  $E \geq 0$ .

### 4 Appendix: The helicity $\pm 1$ sector

We consider here the remaining, helicity 1, subspace involving only the “ $T_i$ ” parts of  $(\pi, h)$  in (1). Clearly, neither  $h G_L(h)$  nor the  $R_0$  constraint involve  $h_{T,i}$ ; only the mass term does: it contains  $\frac{1}{2} m^2 (h_{T,i})^2$ . Its  $\pi$  sector involves  $(\pi_{i,j}^T)^2$  as well as the quadratic term  $2 N_i^T \partial_j \pi^{ij}$ . Using the  $\frac{1}{2} m^2 (N_i^T)^2$  from the mass term, we complete the square to leave a net contribution  $m^{-2} (\pi_{i,j}^{ij})^2 \sim 2 m^{-2} (\pi_i^T \nabla^2 \pi_i^T)$  there. At this point, then, dropping the  $T_i$  indices, we find

$$L(T_i) = -2 \pi \dot{h} - \frac{1}{2} [m^2 h^2 + 4 m^{-2} \pi (m^2 - \nabla^2) \pi]; \quad (14)$$

the necessary redefinition is obvious:

$$\pi \rightarrow -\frac{1}{2} m (m^2 - \nabla^2)^{-1/2} \pi \quad h \rightarrow m^{-1} \sqrt{m^2 - \nabla^2} h \quad (15)$$

leads to the desired helicity 1 canonical Lagrangian,

$$L(T_i) \rightarrow \pi \dot{h} - \frac{1}{2} [\pi^2 + h (m^2 - \nabla^2) h]. \quad (16)$$

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